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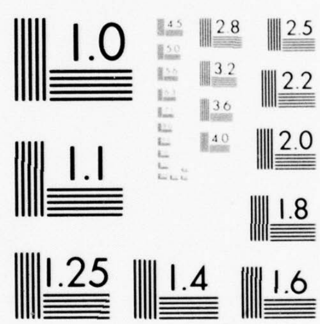

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In conventional approaches to finite element stress analysis accuracy is obtained by fixing the degree p of the approximating polynomial and by allowing the maximum diameter h of elements in the triangulation to approach zero. An alternate approach is to fix the triangulation and to increase the degrees of approximating polynomials in those elements where more accuracy it is necessary to have a family of finite elements of arbitrary polynomial degree p with the property that as much information as possible can be retained from the p^{th} degree approximation when computing the (p+1) st degree approximation. (continued)

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20. Abstract (continued)

Comparison of the optimal beam (a variable-section beam) with a reference beam (a constant-section beam) shows that the weight reduction depends strongly on the frequency parameter beta. This weight reduction is negligible for beta approaching 0, is 11.3 per cent for beta = 1, is 55.3 per cent for beta = 1.4, and approaches 100 per cent for beta approaching 90 degrees.

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The Constraint Method for Solid Finite Elements

Annual Technical Report, October 1, 1977 - September 30, 1978

by

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Professor of Applied Mathematics and Systems Science
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2. Current and Anticipated Results

The ultimate objective of our research is to provide algorithms, and, by using experimental digital computer programs which implement the constraint method for three dimensional analysis, to demonstrate the effectiveness of the method in three dimensions. Since the approaches to be used in deriving these algorithms are based on similar approaches for two dimensional analyses, part of the grant period has been spent in completing the formulation of algorithms in two dimensions so as to have a firm foundation for the more complex three dimensional work. Also in response to a suggestion by Dr. V. B. Venkayya of the Air Force Flight Dynamics Laboratory at Wright-Patterson AFB, the plate bending element of the Constraint Method is being developed further. Results in the first three areas described below are for two dimensional problems in linear stress analysis; results in the fourth area are for three dimensional linear stress analysis.

2.1 C^0 Displacement Fields (plane elasticity)

2.1.1 Hierarchical Elements and Precomputed Arrays

It has been shown previously by others that elemental arrays may be efficiently generated through the use of "precomputed" arrays - - that is arrays which are computed once, stored on permanent file, and then reused in all subsequent applications of the program. The new work done by the principal investigator and his collaborators has two objectives: the first is to show how the hierarchal C^0 elements (described in 1.2) for a quadratic functional may be formulated using precomputed arrays thus yielding a finite element technique which is especially suited to problems with local rapid variation of the function to be approximated. In particular, formulas for two-dimensional (hierarchal) element arrays for arbitrary polynomial order are derived, based on precomputed arrays. The second objective is to apply the combined approach of hierarchal elements and precomputed arrays to decide if a computed result has "converged". A common practice in finite element analysis is to solve a problem several times using successively required meshes i.e. to apply the procedure for h-convergence. If successive analyses agree then it is usually assumed that the finite element approximation is accurate. This procedure can be computationally expensive when several highly refined meshes are used. An alternative procedure is to use p-convergence, which, as pointed out in 1.1, has a faster rate of convergence to the true displacements. The computational effectiveness of the p-convergence procedure is demonstrated numerically using hierarchal elements and precomputed arrays.

Detailed formulas are given for calculation of stiffness matrices and for calculation of polynomial coefficients from nodal variables. Hierarchal nodal variables are presented together with some of the favorable consequences of using hierarchal nodal variables. Computation times for stiffness matrices are given in terms of equivalent time units (e.t.u.) for different methods. It has been demonstrated in [11] that the constraint method allows many more degrees of freedom than other methods do, for the same computer cost.

2.1.2 Singularity Functions used in Linear Elastic Fracture Mechanics

Poor computational efficiency has generally been observed when employing conventional finite elements near a crack top. Therefore, in [17], the behavior of a rational type singularity function which represents an order of $r^{-\frac{1}{2}}$ type stress singularity has been investigated, when used in conjunction with hierarchical C^0 elements, in an effort to improve efficiency. An internal mode of the form

$$\phi(L_1, L_2, L_3) = \frac{L_1 L_2 L_3}{(L_2 + L_3)^{3/2}}$$

is an approximation in triangular coordinates to an $r^{-\frac{1}{2}}$ type stress singularity at the vertix $L_1 = 1$. Using techniques similar to those developed by us in [5], it is possible to integrate derivatives of the singularity function ϕ over triangles, explicitly, and then to employ precomputed arrays to compute elemental stiffness matrices. The p-convergent procedure can then be applied to determine the strain energy U, the strain energy release rate G and crack opening displacements.

To determine the effect of adding the singularity function ϕ to a polynomial basis we have taken the edge cracked panel shown in figure 6. Because of symmetry only one quarter of the panel is modeled with the three different triangulations shown in figure 6. The finite element solution is obtained for each of the triangulations first with only polynomials in the displacement field, and then with the addition of the singularity function in elements meeting at the crack top. The results are plotted in figure 6 and figure 7 and are given in detail in Table 2 and Table 3. It is evident that there is an improvement in the estimates of the strain energy U and the strain energy release rate G but that this improvement is not appreciable compared to the results already obtained by increasing the polynomial order p. Thus, the mode of p-convergence appears to be more important in achieving accuracy than the addition of a singularity function to the basis.

We have also considered the case of a centrally cracked panel shown in figure 8 modeled with five elements in one quarter of the panel. The crack opening displacement (COD) and stresses are computed with the singularity function included in the approximating displacement field and the results are shown in table 4. In table 4, δ is the COD obtained by polynomial approximation above, δ^* is the COD obtained by including a singularity function and ϵ is a measure of component defined as

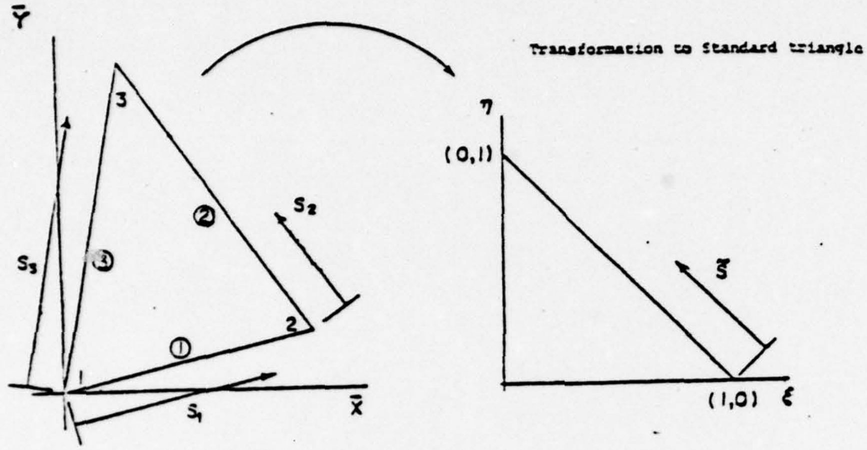
$$\epsilon = \frac{\delta^* - \delta}{\delta} \times 100 .$$

It is clear from table 4 that inclusion of the singularity function in the p-convergent procedure leads to substantial improvement in the displacement near the crack.

More details and other cases which have been studied are presented in [17]. In all cases the case of the constraint method in the mode of p convergence leads to major computational advantages.

2.2 Coupled C^0 and C^1 Displacement Fields

In this work the results of Kratochvil et al in [18] are generalized to problems with three independent displacement fields. An essential aspect of this approach is to transform a triangular element T in the x-y plane into a standard triangle $\tilde{T}$ with vertices at the origin and at a unit distance along the horizontal and vertical axes. Such a transformation is shown on the next page where ξ and η represent coordinates in the plane of the standard triangle and $\bar{x}$ and $\bar{y}$ are local coordinates for the element with the first vertex of the element coinciding with the origin. The other two vertices and also the three edges are numbered in counter-clockwise order as shown.

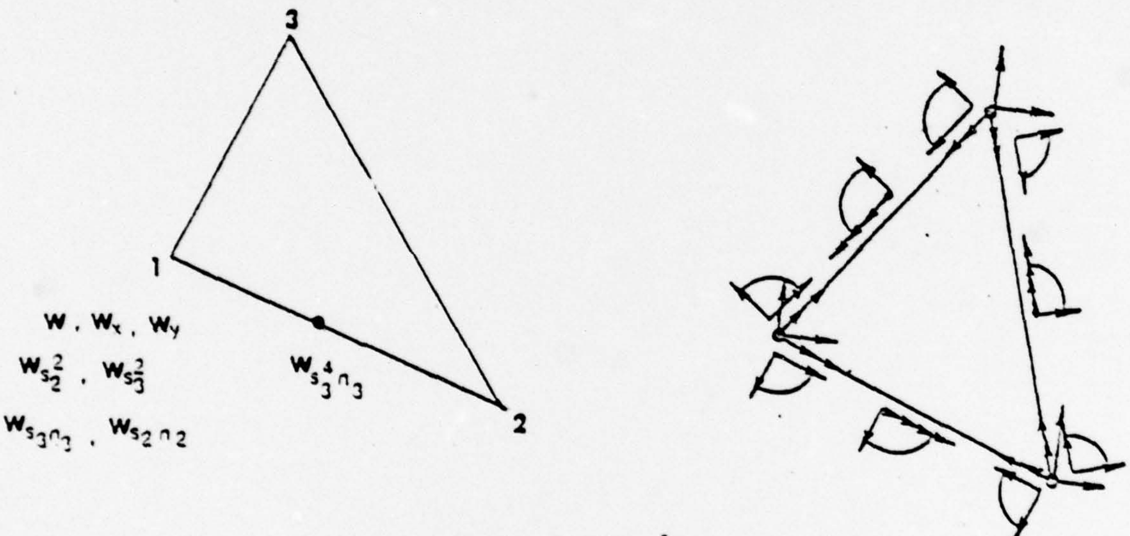


The justification for using the standard triangle is that integrations and matrix inversions are performed with respect to the standard triangle. Thus they need be done only once and the results are stored and then used in all future applications of the program. Computation of the element stiffness matrix is thus reduced to computing a linear combination of a small number of precomputed matrices followed by pre- and post-multiplication by block diagonal matrices. The number of precomputed matrices which must be stored are considerably reduced by choosing hierarchal nodal variables. Further details are given in [12].

2.3 <u>C^1 Displacement Fields (plate bending problems)</u>

It was suggested by Dr. V. B. Venkayya (of the Analysis and Optimization Group, Structures Division, Air Force Flight Dynamics Laboratory, Wright-Patterson Air Force Base) in a letter dated 12 December 1975 that further development of plate bending elements would be useful in order to realize the full potential of the constraint method. Accordingly, a sophisticated plate bending element, incorporating a complete $p^{\underline{th}}$ order polynomial with $p \geq 5$ and corrective rational functions, has been formulated and is now being programmed and tested. We now describe some of this work.

It is well known (see [19], for example) that exactly conforming (even at vertices) C^1 displacement fields cannot be formed merely by freely assembling finite elements. There are certain additional constraint equations which must be satisfied at vertices. The simplest form of these constraint equations has been given by Peano in [4], where a specially devised assembly procedure is presented which automatically enforces the constraint equations. An alternative method for enforcing exact conformity when using displacement fields of arbitrary polynomial order p is by supplementing $p^{\underline{th}}$ order polynomials with newly constructed corrective rational functions. This destroys the analytic character of the approximation at the vertices but permits free assembly of elements without enforcing constraints [4]. An algorithm has been developed for a C^1 (exactly) conforming triangular element which contains complete polynomials of order $p \geq 5$ and corrective rational functions. A typical element of order $p = 5$ together with nodal variables is shown below.



Hierarchic Quintic C^1 Element

The quintic C^1 element has 24 independent nodal variables. The shape functions for these nodal variables are given in Table 5. The shape functions for second order tangential-normal derivatives are rational functions. It is important to observe that although rational functions are used in the basis, all terms which appear in the elemental stiffness matrix can be integrated explicitly without recourse to numerical quadrature. This was proved in [5]. An algorithm based on a hierarchical family of C^1 elements using corrective rational functions has been programmed and is now being tested on numerical examples. We now give one such example.

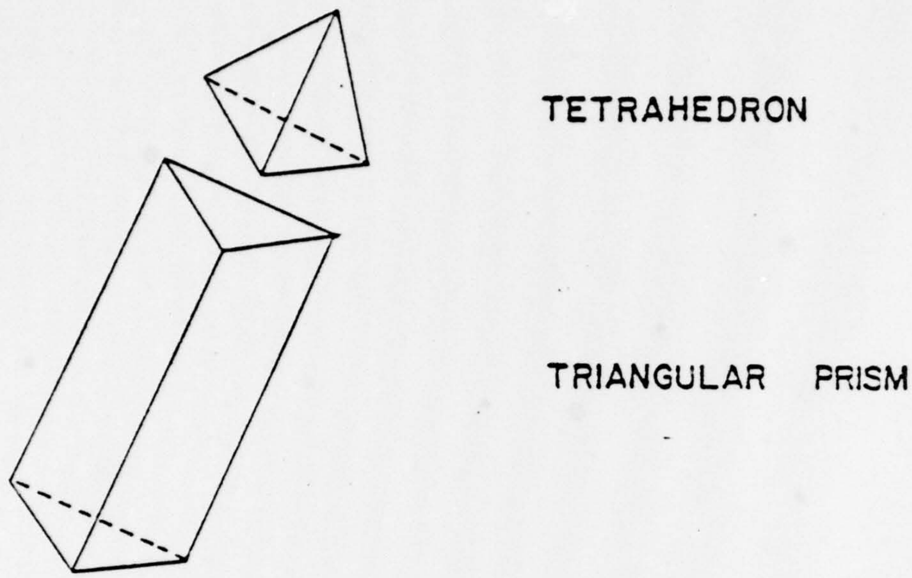
2.3.1 An Example: A Simply Supported Square Plate under Uniform Loading

Results obtained by using the hierarchical quintic C^1 element are compared with those obtained by Cowper in [20], by Caramanlian et al in [21] and by Tsai in [22]. Two different types of triangulation are shown in Figure 9: the Q-arrangement and the P-arrangement. In Table 6 the results for central deflection, central bending moments and for the strain energy are compared for different elemental arrangements and for different C^1 elements. It is seen that for the Q-arrangement the strain energy obtained by using the Constraint Method is an order of magnitude more accurate than that obtained by Cowper, and for the p-arrangement it is two orders of magnitude more accurate. This is consistent with the order of magnitude improvements that have resulted in many problems by using the constraint method.

2.4. Hierarchical Families of Complete Conforming Solid Elements of Various Shapes and Arbitrary Order

In [4] a table of canonical basis functions for a triangular (two dimensional) element was presented, using natural coordinates. This table has been generalized to include a canonical basis for a tetrahedral (three dimensional) hierarchic family. Using this table shape functions are generated for tetrahedral C^0 elements and their corresponding nodal variables. Table 7 gives the nodal variables and shape functions for the first four hierarchic Tetrahedral C^0 elements.

A hierarchic family of rectangular C^0 elements has been developed and this family is used to generate a hierarchic family of C^0 brick elements of arbitrary polynomial order. By using a combination of the hierarchic triangular and rectangular families, we have also constructed a hierarchic family of triangular prismatic elements. These prismatic elements, which have also have arbitrary polynomial order, can be made to join continuously to tetrahedral elements, so that pointed prismatic geometries can now be easily approximated (see the figure on the next page).



2.5 Papers for Publication in Journals and Presentation at Conferences

The various aspects of the work described in 2.1 - 2.4 are in different stages of development. Listed below are papers either already accepted for publication or in preparation, and papers already presented or to be presented at conferences.

Published Papers:

1). "Hierarchal Finite Elements and Precomputed Arrays", by Mark P. Rossow and I. Norman Katz, (to appear in Int. J. for Num. Method in Engr.).

2). "Nodal Variables for Conforming Finite Elements of Arbitrary Polynomial Order", by I. Norman Katz and Mark P. Rossow, (to appear in Computers and Mathematics, with Applications).

3). "A Hierarchic Family of Complete, Conforming C^1 Triangular Elements, for Plate Bending", by I. Norman Katz and Barna A. Szabo, (in preparation).

4). "Hierarchic Families of Complete Conforming Solid Finite Elements of Various Shapes", by I. Norman Katz, (in preparation).

5). "P-convergent Finite Element Approximations in Linear Elastic Fracture Mechanics", by Anil K. Mehta (doctoral dissertation) Department of Civil Engineering, Washington University (1978).

Presented Papers:

6). "Hierarchical Approximation in Finite Element Analysis", by I. Norman Katz, International Symposium on Innovative Numerical Analysis in Applied Engineering Science, Versailles, France, May 23 - 27, 1977.

7). "Efficient Generation of Hierarchal Finite Elements Through the Use of Precomputed Arrays", by M. P. Rossow and I. N. Katz, Second Annual ASCE Engineering Mechanics Division Specialty Conference, North Carolina State University, Raleigh, NC May 23 - 25, 1977.

8). "C^1 Triangular Elements of Arbitrary Polynomial Order Containing Corrective Rational Functions", by I. Norman Katz, SIAM 1977 National Meeting, Philadelphia, PA, June 13 - 15, 1977.

9). "Hierarchical Complete Conforming Tetrahedral Elements of Arbitrary Polynomial Order", by I. Norman Katz, presented at SIAM 1977 Fall Meeting, Albequerque, NM, October 31 - November 2, 1977.

10). "A Hierarchical Family of Complete Conforming Prismatic Finite Elements of Arbitrary Polynomial Order", by I. Norman Katz to be presented at SIAM 1978 National Meeting, Madison, WI, May 24-26, 1978.

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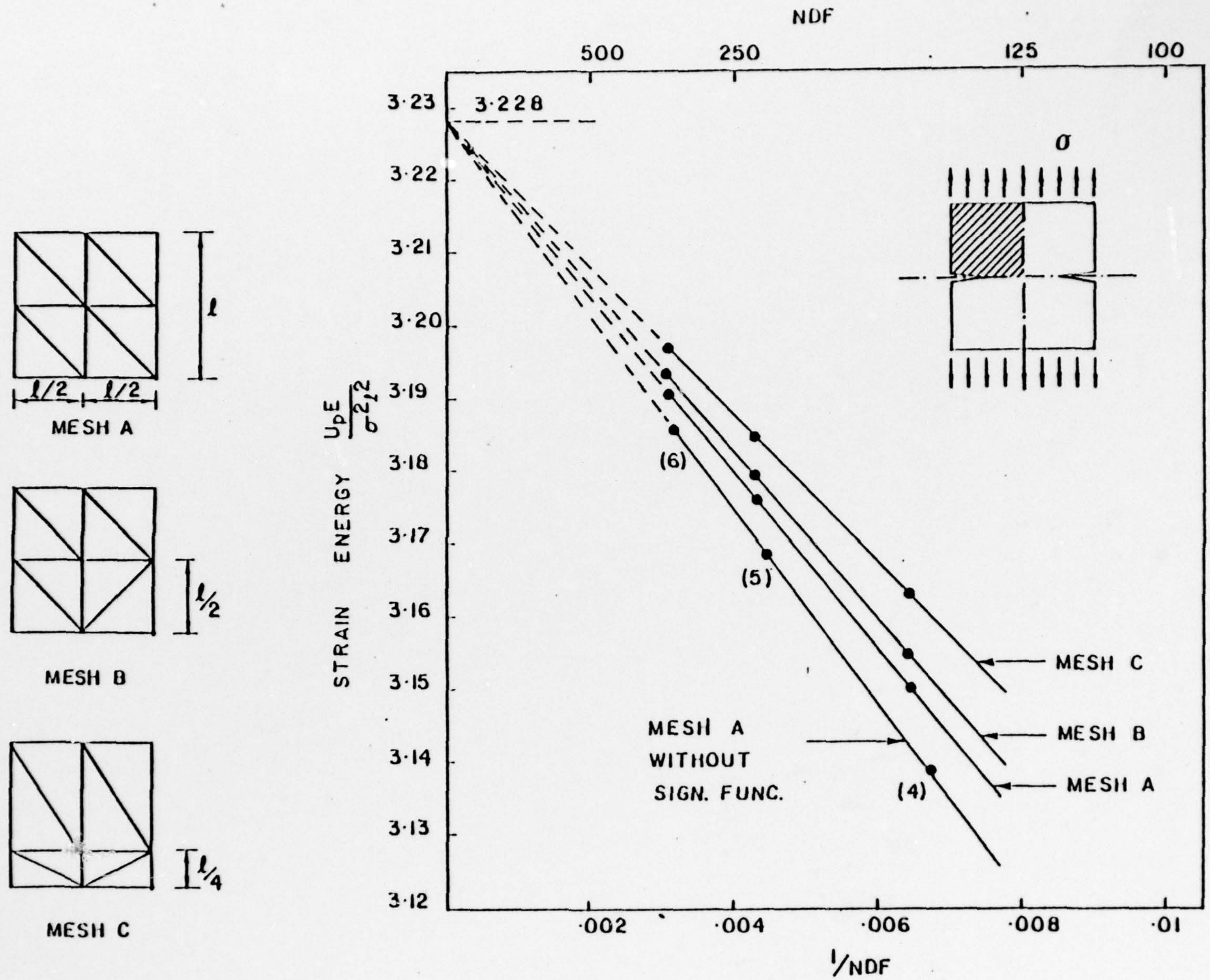
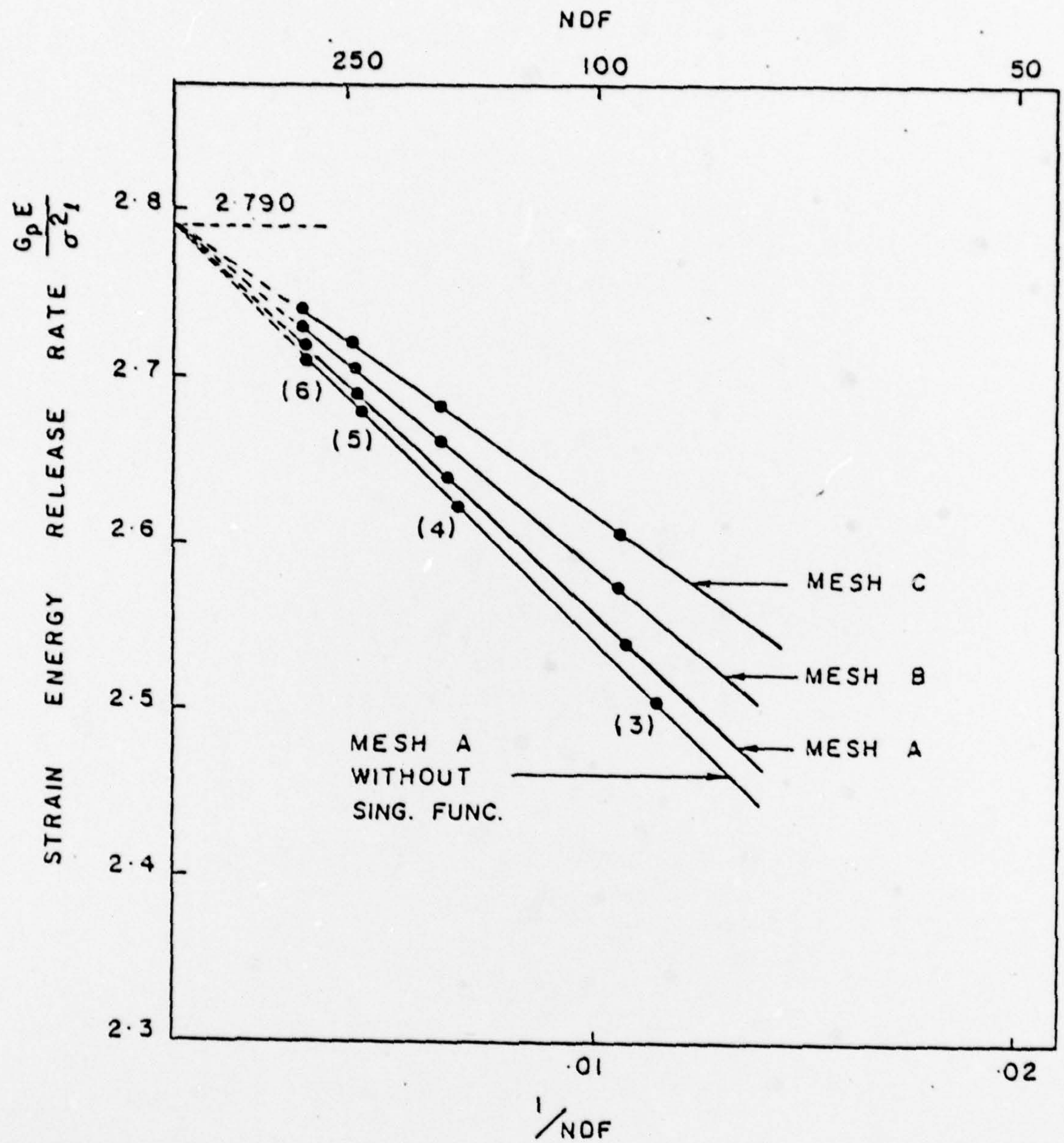
[11] Rossow, M. P. and Katz, I. N., "Hierarchical Finite Elements and Precomputed Arrays", to appear in Int. J. Num. Meth. Engng.

[12] Katz, I. N. and Rossow, M. P., "Nodal Variables for Complete Conforming Finite Elements of Arbitrary Polynomial Order", to appear in Computers and Mathematics with Applications.



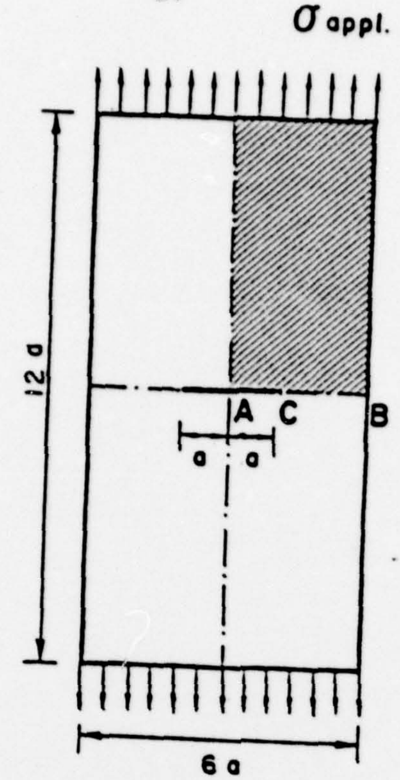
Strain Energy in an edge-cracked panel using a singularity function

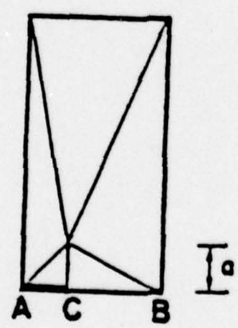
Figure 6



Strain Energy Release Rate in an edge-cracked panel
using a singularity function

Figure 7



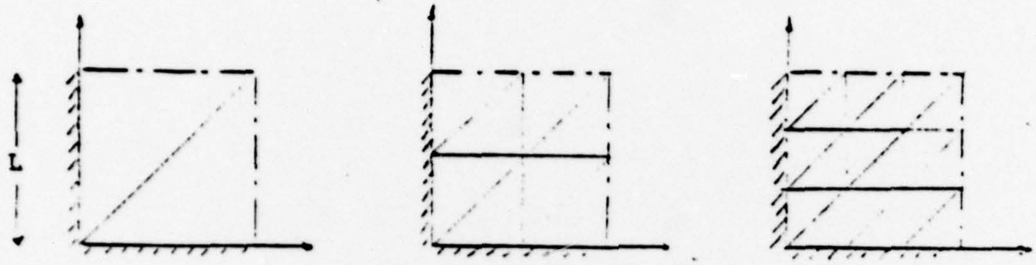


Centrally-cracked panel

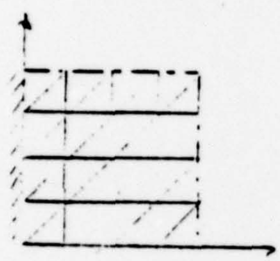
Figure 8

Arrangement of Finite Elements in a Square Plate
under a uniform load

Q-Arrangement

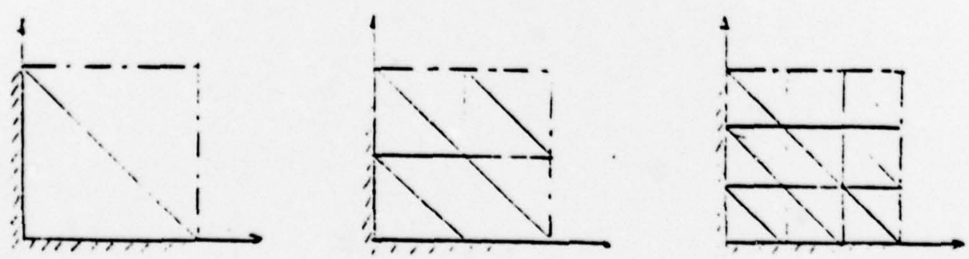


N=2 N=4 N=6



N=8

P-Arrangement



N=2 N=4 N=6

Figure 9

TABLE 2

Comparison of Strain Energy Approximations
for Various Polynomial Orders and Mesh Divisions
With and Without the Inclusion of Singularity Mode

p	Mesh	Strain Energy $U_pE/\sigma^2\ell^2$	Error (%)	$U_p^*E/\sigma^2\ell$	Error (%)
	A	2.9582796	8.356	2.9815737	7.634
2	B	2.9778692	7.749	2.9909923	7.342
	C	2.9876714	7.445	2.9957897	7.194
	A	3.0809992	4.554	3.0976153	4.039
3	B	3.0961462	4.085	3.1050224	3.810
	C	3.1124021	3.581	3.1172042	3.432
	A	3.1391180	2.753	3.1503687	2.405
4	B	3.1488200	2.453	3.1547416	2.269
	C	3.1603298	2.096	3.1635274	1.997
	A	3.1683829	1.847	3.1761374	1.607
5	B	3.1751263	1.638	3.1791517	1.513
	C	3.1826309	1.405	3.1848030	1.338
	A	3.1850978	1.329	3.1907447	1.154
6	B	3.1899564	1.179	3.1928865	1.088
	C	3.1955412	1.001	3.1971365	0.956

*Singularity function is included in the elements meeting at the crack tip.

TABLE 3

Comparison of Strain Energy Release Rate Approximations
for Various Polynomial Orders and Mesh Divisions
With and Without the Inclusion of Singularity Mode

p	Mesh	Strain Energy Release Rate $G_p E/\sigma^2 \ell$	Error (%)	$G_p^* E/\sigma^2 \ell$	Error (%)
	A	2.2735840	18.510	2.3195304	16.862
2	B	2.3548444	15.597	2.3668902	15.165
	C	2.3926472	14.242	2.4012298	13.934
	A	2.5083682	10.094	2.5375678	9.048
3	B	2.5660932	8.025	2.5726580	7.790
	C	2.5995090	6.828	2.6035822	6.682
	A	2.6206222	6.071	2.6405090	5.358
4	B	2.6592740	4.686	2.6636310	4.529
	C	2.6830336	3.834	2.6859452	3.729
	A	2.6752748	4.112	2.6883926	3.642
5	B	2.7015856	3.169	2.7041712	3.076
	C	2.7184304	2.565	2.7203946	2.495
	A	2.7074112	2.960	2.7170000	2.616
6	B	2.7262540	2.285	2.7282638	2.213
	C	2.7391374	1.823	2.7405716	1.772

*Singularity function is included in the elements meeting at the crack tip

TABLE 4

Comparison of the Crack Opening Displacements (COD)
for Various Polynomial Orders
With and Without the Inclusion of Singularity Modes

p	COD	x/a, the distance along the crack surface from the center of the crack									
		0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	δ	0.4825	0.4960	0.4958	0.4819	0.4542	0.4125	0.3577	0.2889	0.2030	0.1100
2	δ^*	0.5042	0.5200	0.5212	0.5076	0.4792	0.4362	0.3784	0.3059	0.2137	0.1167
	ϵ	4.5	4.8	5.1	5.3	5.5	5.7	5.8	5.9	7.7	6.1
	δ	0.6326	0.6204	0.6078	0.5908	0.5652	0.5268	0.4716	0.3955	0.2942	0.1638
3	δ^*	0.6443	0.6304	0.6178	0.6040	0.5814	0.5460	0.4926	0.4163	0.3121	0.1750
	ϵ	1.8	1.6	1.6	2.2	2.9	3.6	4.4	5.3	6.1	6.8
	δ	0.6795	0.6694	0.6522	0.6304	0.6036	0.5690	0.5212	0.4523	0.3517	0.2061
4	δ^*	0.6865	0.6771	0.6589	0.6363	0.6101	0.5779	0.5337	0.4683	0.3689	0.2293
	ϵ	1.0	1.1	1.0	1.0	1.1	1.6	2.4	3.5	4.8	6.4
	δ	0.6910	0.6840	0.6727	0.6527	0.6237	0.5865	0.5397	0.4772	0.3831	0.2388
5	δ^*	0.6959	0.6889	0.6784	0.6581	0.6281	0.5904	0.5451	0.4863	0.3985	0.2521
	ϵ	0.7	0.7	0.8	0.8	0.7	0.7	1.0	1.9	3.5	5.6
	δ	0.6978	0.6923	0.6800	0.6623	0.6359	0.5983	0.5490	0.4868	0.4016	0.2622
6	δ^*	0.7013	0.6958	0.6834	0.6664	0.6404	0.6019	0.5520	0.4911	0.4103	0.2744
	ϵ	0.5	0.5	0.5	0.6	0.7	0.6	0.5	0.9	2.2	4.7
	δ	0.7012	0.6985	0.6866	0.6667	0.6403	0.6048	0.5556	0.4922	0.4091	0.2732
7	δ^*	0.7039	0.7014	0.6893	0.6691	0.6434	0.6084	0.5590	0.4945	0.4139	0.2884
	ϵ	0.4	0.4	0.4	0.4	0.5	0.6	0.6	0.5	1.2	3.7

δ without singularity functions

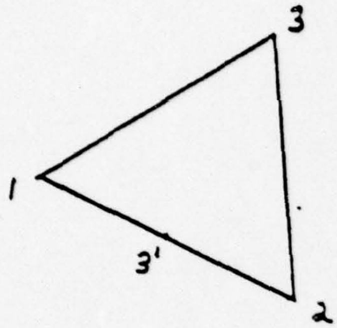
δ^* with singularity functions in the two elements meeting at the crack tip

ϵ is the improvement quotiant with singularity functions and is given by $\frac{\delta^*-\delta}{\delta} \times 100$

TABLE 5

SHAPE FUNCTIONS FOR FIFTH ORDER C^1 ELEMENT

CONTAINING CORRECTIVE RATIONAL FUNCTIONS



$$\mu_3 = \frac{\ell_2^2 - \ell_1^2}{\ell_3^2}$$

$$\mu_2 = \frac{\ell_1^2 - \ell_3^2}{\ell_2^2}$$

NODAL VARIABLE	SHAPE FUNCTION
$\frac{\partial^5 w}{\partial s_3^4 \partial n_3}(3')$	$N_{21} = L_1^2 L_2^2 L_3$
$\frac{\partial^2 w}{\partial s_3^2}(1)$	$N_4 = \frac{1}{2} L_1^2 L_2^2 (1-L_2) - \frac{1}{4}(1+5\mu_3) N_{21} + \frac{1}{2}(1+\mu_3)\eta_1$
$\frac{\partial^2 w}{\partial s_2^2}(1)$	$N_6 = \frac{1}{2} L_1^2 L_3^2 (1-L_3) - \frac{1}{4}(1-5\mu_2) N_{20} + \frac{1}{2}(1-\mu_2)\eta_2$ $(N_{20} = L_1^2 L_2 L_3^2)$
$-\frac{\partial^2 w}{\partial s_3 \partial s_2}(1)$	$N_5 = L_1^2 L_2 L_3$
$\frac{\partial w}{\partial s_3}(1)$	$N_2 = L_1^2 L_2(1-L_2) + 2N_5 + 6N_4$
$-\frac{\partial w}{\partial s_2}(1)$	$N_3 = L_1^2 L_3(1-L_3) + 2N_5 + 6N_6$
$w(1)$	$N_1 = L_1^3 + 3(N_2+N_3) - 6(N_4+N_5+N_6)$
$\frac{\partial^2 w}{\partial s_3 \partial n_3}(1)$	$\eta_1 = \frac{L_1^2 L_2^2 L_3}{L_2 + L_3}$ CORRECTIVE RATIONAL FUNCTIONS
$-\frac{\partial^2 w}{\partial s_2 \partial n_2}(1)$	$\eta_2 = \frac{L_1^2 L_2 L_3^2}{L_2 + L_3}$

TABLE 6

SIMPLY SUPPORTED SQUARE PLATE

UNIFORM LOAD

(5^{th} order Approximation)

	N	P/Q	Constraint Method	Cowper	Caramanlian	Tsai
CENTRAL DEFLECTION $10^3 W_c D/q_0 L^4$	2	Q	4.060210	4.0609374	4.060236	4.0597
		P	4.059696	4.0684849	4.069917	
	4	Q	4.062342	4.0623473	4.062323	4.0628
		P	4.062483	4.0627265	4.06250	
	6	Q	4.0623498	4.0623517		
		P	4.0623503	4.0623898		
	8	Q	4.0623522	4.0623524		

EXACT SOLUTION: 4.0623527

	N	P/Q	Constraint Method	Cowper	Caramanlian	Tsai
CENTRAL BENDING MOMENTS $10^2 M_{xx}/q_0 L^2$	2	Q	4.68094	4.70261	4.6579	4.6127
		P	4.84074	4.81530	4.8568	
	4	Q	4.78697	4.78259	4.7816	4.8338
		P	4.79074	4.79083	4.7924	
	6	Q	4.787436	4.787621		
		P	4.789028	4.788972		
	8	Q	4.788329	4.788419		

EXACT SOLUTION: 4.788683

TABLE 6 (continued)

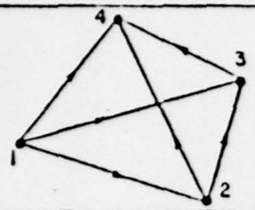
STRAIN ENERGY

$10^4\ UD/q_0L^6$

N	P/Q	Cowper	% Error	Constraint Method	% Error
2	Q	8.5099612	3.04×10^{-2}	8.5117189	9.79×10^{-3}
	P	8.4776356	4.10×10^{-2}	8.5124819	8.31×10^{-4}
4	Q	8.5124403	1.32×10^{-3}	8.5125393	1.56×10^{-4}
	P	8.5113962	1.36×10^{-2}	8.5125227	3.51×10^{-4}
6	Q	8.5125386	1.64×10^{-4}	8.5125514	1.41×10^{-5}
	P	8.5124190	1.57×10^{-3}	8.5125501	2.94×10^{-5}
8	Q	8.5125496	3.52×10^{-5}	8.5125524	2.35×10^{-6}

EXACTION SOLUTION: 8.5125526

TABLE 7

	NODAL VARIABLES AND SHAPE FUNCTIONS FOR THE FIRST FOUR HIERARCHIC TETRAHEDRAL C^0 ELEMENTS
1. Linear	4 terms. Values of the approximating function at the vertices. $N_1 = L_1,\ N_2 = L_2,\ N_3 = L_3,\ N_4 = L_4$
2. Quadratic	6 additional terms: Second derivatives at the midpoints of edges. Normalizing factor: -1/ $N_5 = L_1L_2,\ N_6 = L_2L_3,\ \ldots\ N_{10} = L_4L_1$
3. Cubic	10 additional terms: Six third derivatives at the midpoints of edges. Normalizing factor: 1/12. $N_{11} = L_1^2L_2 - L_1L_2^2,\ N_{12} = L_2^2L_3 - L_2L_3^2,\ \ldots\ N_{16} = L_1^2L_4 - L_1L_4^2$ Four face modes: $N_{17} = L_1L_2L_3,\ N_{18} = L_1L_2L_4,\ N_{19} = L_1L_3L_4,\ N_{20} = L_2L_3L_4$
4. Quartic	15 additional terms: Six fourth derivatives at the midpoints of edges. Normalizing factor: - 1/48: $N_{21} = L_1^3L_2 + L_1L_2^3,\ N_{22} = L_2^3L_3 + L_2L_3^3,\ \ldots\ N_{26} = L_1^3L_4 + L_1L_4^3$ Eight face modes: $N_{27} = L_1^2L_2L_3,\ N_{28} = L_1L_2^2L_3,\ N_{29} = L_1^2L_2L_4,\ N_{30} = L_1L_2^2L_4,\ N_{31} = L_1^2L_3L_4$ $N_{32} = L_1L_3^2L_4,\ N_{33} = L_2^2L_3L_4,\ N_{34} = L_2L_3^2L_4$ One internal mode: $N_{35} = L_1L_2L_3L_4$

LIST OF PROFESSIONAL PERSONNEL ASSOCIATED WITH THE RESEARCH EFFORT

1) I. Norman Katz, Professor of Applied Mathematics and Systems Science, Washington University, St. Louis, MO 63130.

2) Barna A. Szabo, A. P. Greensfelder Professor of Civil Engineering, Washington University, St. Louis, MO 63130.

3) Mark P. Rossow, Associate Professor of Civil Engineering, Washington University, St. Louis, MO 63130.

4) Mai Sen Chang, Graduate Student, Department of Systems Science and Mathematics, Washington University, St. Louis, MO 63130.

5) Anil K. Mehta, Graduate Student, Department of Civil Engineering, Washington University, St. Louis, MO 63130. (D.Sc May 1978, Thesis Title: "p-convergent finite element approximation in linear elastic fracture mechanics")

INTERACTIONS

On October 19, 1977, the Principal Investigator, I. Norman Katz, presented an invited talk at the Expository Seminar Series of the Applied Mathematics Division of the U.S. National Bureau of Standards in Gaithersburg, MD. An abstract of the talk, based on current research, is enclosed.

National Bureau of Standards
Applied Mathematics Division

Expository Seminar Series

OCTOBER MEETING

Date & Time:	Wednesday, October 19, 1977 ------ 11:00 A.M. (Coffee Social -------- 10:45 A.M.)
Place:	Technology Building, Room B27 National Bureau of Standards Gaithersburg, Maryland 20760
Speaker:	Professor I. Norman Katz Department of Systems Science and Mathematics Washington University St. Louis, Missouri
Title of Talk:	"The Constraint Method for Finite Element Stress Analysis"

Abstract:

In conventional approaches to finite element stress analysis accuracy is obtained by fixing the degree p of the approximating polynomial and by allowing the maximum diameter h of elements in the triangulation to approach zero. An alternate approach is to fix the triangulation and to increase the degrees of approximating polynomials in those elements where more accuracy is required. In order to implement the second approach efficiently it is necessary to have a family of finite elements of arbitrary polynomial degree p with the property that as much information as possible can be retained from the p^{th} degree approximation when computing the $(p+1)$st degree approximation. Such a HIERARCHIC family has been formulated with $p \geq 2$ for problems in plane stress analysis and with $p \geq 5$ for problems in plate bending. The family is described and numerical examples are presented which illustrate the efficiency of the new method.

Biographical Sketch:

Professor Katz received his Ph.D. in Mathematics from M.I.T. in 1959. He has worked at AVCO/Research and Advanced Development in Wilmington, Massachusetts where he became Manager of the Mathematics Department. Since 1967 he has been at Washington University in St. Louis. His research has been in numerical analysis, ordinary and partial differential equations, finite element methods, optimal facility location and biomathematics.

All interested are invited to attend.

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